A NOTE ON INNER QUASIDIAGONAL C*-ALGEBRAS

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Abstract. In the paper, we give two new characterizations of separable inner quasidiagonal C*-algebras. Base on these characterizations, we show that a unital full free product of two inner quasidiagonal C*-algebras is inner quasidiagonal again. As an application, we show that a unital full free product of two inner quasidiagoanl C*-algebras with amalgmation over a full matrix algebra is inner quasidiagonal. Meanwhile, we conclude that a unital full free product of two AF algebras with amalgamation over a finite-dimensional C*-algebra is inner quasidiagonal if there are faithful tracial states on each of these two AF algebras such that the restrictions on the common subalgebra agree.

1. Introduction

Quasidiagonal (QD) C*-algebras have now been studied for more than 30 years. Voiculescu [17] give a characterization of quasidiagonal C*-algebras as following:

Definition 1. *A C*-algebra $\mathcal{A}$ is quasidiagonal if, for every $x_1, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $P \in \mathcal{B}(\mathcal{H})$ such that $\|P\pi(x_i) - \pi(x_i)P\| < \varepsilon, \|P\pi(x_i)P\| > \|x_i\| - \varepsilon$ for $1 \le i \le n$.*

Voiculescu showed that $\mathcal{A}$ is QD if and only if $\pi(\mathcal{A})$ is a quasidiagonal set of operators for a faithful essential representation π of $\mathcal{A}$. In [4], we know that all separable QD C*-algebras are Blackadar and Kirchberg's MF algebras. It is well known that the reduced free group C*-algebra $C_r^*(F_2)$ is not QD. Haagerup and Thorbjϕrnsen showed that $C_r^*(F_2)$ is MF [11]. This implies that the family of all separable QD C*-algebras are strictly contained in the set of MF C*-algebras.

The concept of MF algebras was first introduced by Blackadar and Kirchberg in [4]. Many properties of MF algebras were discussed in [4]. In the same article, Blackadar and Kirchberg study NF algebras and strong NF algebras as well. A separable C*-algebra is a strong NF algebra if it can be written as a generalized inductive limit of a sequential inductive system of finite-dimensional C*-algebras in which the connecting maps are complete order embedding and asymptotically multiplicative in the sense of [4]. An NF algebra is a C*-algebra which can be written as the generalized inductive limit of such system, where the connecting maps are only required to be completely positive contractions. It was shown that a separable C*-algebra is an NF algebra if and only if it is nuclear and quasidiagonal. Whether the class of NF algebra is distinct from the class of strong NF algebras?

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For solving this question, Blackadar and Kirchberg introduce the concept of inner quasidiagonal by slightly modifying Voiculescu's characterization of quasidiagonal C*-algebras:

Definition 2. *([5]) A C*-algebra $\mathcal{A}$ is inner quasidiagonal if, for every $x_1, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $P \in \pi(\mathcal{A})''$ such that $\|P\pi(x_i) - \pi(x_i)P\| < \varepsilon, \|P\pi(x_i)P\| > \|x_i\| - \varepsilon$ for $1 \leq i \leq n$.*

It was shown that a separable C*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal [5]. Blackadar and Kirchberg also gave examples of separable nuclear C*-algebras which are quasidiagonal but not inner quasidiagonal, hence of NF algebras which are not strong NF. Therefore, the preceding question has been solved.

In this note, we are interested in the question of whether the unital full free products of inner QD C*-algebras are inner QD again. Note that every RFD C*-algebra is inner QD [5]. We have known that a unital full free products of two RFD C*-algebras is RFD [16]. Similar result holds for unital QD C*-algebras [2]. Based on these results and the relationship among RFD C*-algebras, inner QD C*-algebras and QD C*-algebras, it is natural to ask whether the same things will happen when we consider inner QD C*-algebras. In this note we will show that a unital full free product of two unital inner QD C*-algebra is inner again. As an application, we will consider the unital full free products of two inner QD C*-algebras with amalgamation over finite-dimensional C*-algebras.

All C*-algebras in this note are unital and separable. A brief overview of this paper is as follows. In Section 2, we fix some notation and give two new characterizations of inner QD C*-algebras. Section 3 is devoted to results on the unital full free products of two unital inner QD C*-algebras. We first consider unital full free products of unital inner QD C*-algebras. As an application, we show that a unital full free product of two inner quasidiagonal C*-algebras with amalgamation over a full matrix algebra is inner quasidiagonal. Meanwhile, we conclude that a unital full free product of two AF algebras with amalgamation over a finite-dimensional C*-algebra is inner quasidiagonal if there are faithful tracial states on each of these two AF algebras such that the restrictions on the common subalgebra agree.

2. Inner Quasidiagonal C*-algebras

We denote the set of all bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Suppose $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ is a sequence of complex matrix algebras. We introduce the C*-direct product $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ of $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ as follows:

$$\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \{(Y_n)_{n=1}^{\infty} \mid \forall\, n \geq 1,\ Y_n \in \mathcal{M}_{k_n}(C) \ \text{ and } \ \|(Y_n)_{n=1}^{\infty}\| = \sup_{n\geq 1} \|Y_n\| < \infty\}.$$

Furthermore, we can introduce a norm-closed two sided ideal in $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ as follows:

$$\sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \left\{ (Y_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) : \lim_{n\to\infty} \|Y_n\| = 0 \right\}.$$

Let π be the quotient map from $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ to $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})/\sum\limits_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$. Then

$$\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})/\sum_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$$

is a unital C*-algebra. If we denote $\pi\left((Y_n)_{n=1}^{\infty}\right)$ by $[(Y_n)_n]$, then

$$\|[(Y_n)_n]\| = \limsup_{n\to\infty}\|Y_n\| \leq \sup_n\|Y_n\| = \|(Y_n)_n\| \in \prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$$

Recall that a C*-algebra is residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. If a separable C*- algebra $\mathcal{A}$ can be embedded into $\prod\limits_{k}\mathcal{M}_{n_k}(\mathbb{C})/\sum_k\mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, then $\mathcal{A}$ is called an MF algebra. Many properties of MF algebras were discussed in [4]. Note that the family of all RFD C*-algebras is strictly contained in the family of all inner QD C*-algebras, and all QD C*-algebras are MF C*-algebras.

Continuing the study of generalized inductive limits of finite-dimensional C*-algebras, Blackadar and Kirchberg define a refined notion of quasidiagonality for C*-algebras, called inner quasidiagonality. A cleaner alternative definition of inner quasidiagonality can be given using the socle of the bidual.

Definition 3. *If $\mathcal{B}$ is a C*-algebra, then a projection $p\in\mathcal{B}$ is in the socle if $p\mathcal{B}p$ is finite-dimensional. Denote the set of the socle in $\mathcal{B}$ by socle$(\mathcal{B})$*

Theorem 1. *([8]) A separable C*-algebra $\mathcal{A}$ is inner QD if there are projections $p_n\in\mathcal{A}^{**}$ such that*

1. $\|[p_n,a]\|\longrightarrow 0$ *for all* $a\in\mathcal{A}\subseteq\mathcal{A}^{**}$,
2. $\|a\| = \lim\|p_n a p_n\|$ *for all* $a\in\mathcal{A}$ *and*
3. $p_n\in$ *socle*$(\mathcal{A}^{**})$ *for every* n.

Theorem 2. *([5], Proposition 3.7.) Let $\mathcal{A}$ be a separable C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a sequence of irreducible representation $\{\pi_n\}$ of $\mathcal{A}$ on Hilbert space $\mathcal{H}_n$,and finite-rank projection $p_n\in\mathcal{B}(\mathcal{H}_n)$, such that $\|[p_n,\pi_n(x)]\|\longrightarrow 0$ and $\limsup\|p_n\pi_n(x)p_n\| = \|x\|$ for all $x\in\mathcal{A}$.*

The principal shortcoming of the definition of inner QD C*-algebra is that it is often difficult to determine directly whether a C*-algebra is inner QD, the following result for separable case is much easier for checking.

Theorem 3. *([6]) A separable C*-algebra is inner QD if and only if it has a separating family of quasidiagonal irreducible representations.*

Let $\pi:\mathcal{B}(\mathcal{H})\to\mathcal{Q}(\mathcal{H})$ be the canonical mapping onto the Calkin algebra and $\mathcal{A}$ is a unital C*-algebra. Suppose $\varphi:\mathcal{A}\to\mathcal{B}(\mathcal{H})$ is a unital completely positive map then we say that φ is a representation modulo the compacts if $\pi\circ\varphi:\mathcal{A}\to\mathcal{Q}(\mathcal{H})$ is a *-homomorphism. If $\pi\circ\varphi$ is injective then we say that φ is a faithful representation modulo the compacts.

For an MF C*-algebra, we are able to embed it into $\prod\limits_{k}\mathcal{M}_{n_k}(\mathbb{C})/\sum_k\mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$. For an RFD C*-algebra, we can embed it into $\prod\limits_{k}\mathcal{M}_{n_k}(\mathbb{C})$. Meanwhile, for a QD C*-algebra, we can not only embed it into

$\prod\limits_{k}\mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$, but also lift this embedding to a faithful representation into $\prod \mathcal{M}_{k_m}(\mathbb{C})$ modulo the compacts. Whether there is a similar characterization for the inner QD C*-algebras? We will answer this question in the following theorem.

The following lemma is a well-known result about completely positive map. We use c.p. to abbreviate "completely positive", u.c.p. for "unital completely positive" and c.c.p. for "contractive completely positive".

Lemma 1. *(Stinespring) Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be a c.p.. map. Then, there exist a Hilbert space $\widehat{\mathcal{H}}$, a *-representation $\pi_\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ and an operator $V : \mathcal{H} \longrightarrow \widehat{\mathcal{H}}$ such that*

$$\varphi(a) = V^* \pi_\varphi(a) V$$

for every $a \in \mathcal{A}$. In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$.

We call the triplet $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ in preceding lemma a Stinespring dilation of φ. When φ is unital, $V^*V = \varphi(I) = I$, and hence V is an isometry. So in this case we may assume that V is a projection P and $\varphi(a) = P\pi_\varphi(a)|_{\mathcal{H}}$. In general there could be many different Stinespring dilations, but we may always assume that a dilation $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ is minimal in the sense that $\pi_\varphi(\mathcal{A}) V\mathcal{H}$ is dense in $\widehat{\mathcal{H}}$. We know that, under this minimality condition, a Stinespring dilation is unique up to unitary equivalence. Note that if $\left(\pi_\varphi, \widehat{\mathcal{H}}, V\right)$ is minimal Stinespring dilation of $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, then there exists a *-homomorphism $\rho : \varphi(\mathcal{A})' \longrightarrow \pi_\varphi(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}\right)$ such that $\varphi(a)x = V^*\pi_\varphi(a)\rho(x)V$ for every $a \in \mathcal{A}$ and $x \in \varphi(\mathcal{A})'$, it implies that the commutant $\varphi(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{H})$ also lifts to $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$.

Lemma 2. *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \longrightarrow \mathcal{M}_n(\mathbb{C})$ be a surjective u.c.p. map. Suppose $\left(\pi_\varphi, \widehat{\mathcal{H}}, P\right)$ is a minimal Stinespring dilation of φ where P is a projection in $\mathcal{B}\left(\widehat{\mathcal{H}}\right)$. Then the *-homomorphism $\rho : \varphi(\mathcal{A})' \longrightarrow \pi_\varphi(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}\right)$ is unital, i.e.*

$$\rho(\mathbb{C}I) = \mathbb{C}I \subseteq \pi_\varphi(\mathcal{A})'$$

Proof. Since φ is surjective, $\varphi(\mathcal{A})' = \mathbb{C}I$. Note ρ is a *-homomorphism, it is easy to check that

$$(I-P)\rho(\alpha I)P = P\rho(\alpha I)(I-P) = 0.$$

and $(I-P)\rho(I)(I-P)$ is a projection. We know $\left(\pi_\varphi, \widehat{\mathcal{H}}, P\right)$ is a minimal Stinespring dilation, then $\pi_\varphi(\mathcal{A})'$ has no proper projection bigger than P. It implies $(I-P)\rho(I)(I-P) = 0$, i.e. $\rho(I) = I$. Hence $\rho(\mathbb{C}I) = \mathbb{C}I \subseteq \pi_\varphi(\mathcal{A})'$. □

Now, we are ready to give a new characterization of inner QD C*-algebras.

Theorem 4. *Suppose $\mathcal{A}$ is a unital C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a faithful representation modulo compacts $\Phi : \mathcal{A} \longrightarrow \Pi\mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}$ of integers such that the u.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ is surjective for every n and the *-homomorphism*

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_{\varphi_n}(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}_n\right)$$

is surjective where $\left(\pi_{\varphi_n}, \widehat{\mathcal{H}}_n, p_n\right)$ *is a minimal Stinespring dilation of* φ_n.

Proof. ($\Longrightarrow$) Suppose $\mathcal{A}$ is inner QD. Then, by applying Theorem 2, we can find sequences of irreducible representations $\{\pi_n\}$ and finite projections $\{p_n\}$ where $p_n \in \pi_n(\mathcal{A})''$ such that $\Phi : \mathcal{A} \longrightarrow \Pi p_n \pi_n(\mathcal{A}) p_n$ is a faithful representation modulo compacts. Meanwhile, we have $p_n \pi_n(\mathcal{A}) p_n \cong \mathcal{M}_{k_n}(\mathbb{C})$ for some integer k_n and $(\pi_n(\mathcal{A}))' = \mathbb{C}I$ since π_n is irreducible. Define

$$\varphi_n : \mathcal{A} \longrightarrow p_n \pi_n(\mathcal{A}) p_n \cong \mathcal{M}_{k_n}(\mathbb{C})$$

by $\varphi_n(a) = p_n \pi_n(a) p_n$. Then φ_n is u.c.p. and surjective for every n. Note $(\pi_n, \mathcal{H}_n, p_n)$ is a minimal Stinespring dilation of φ_n since π_n is irreducible. Therefore the *-homomorphism

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_n(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{H}_n)$$

is surjective by Lemma 2 and the fact that $\pi(\mathcal{A})' = \mathbb{C}I$.

($\Longleftarrow$) Suppose there is a faithful representation modulo compacts $\Phi : \mathcal{A} \longrightarrow \Pi \mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}$ such that the u.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ is surjective and the *-homomorphism

$$\rho : \varphi_n(\mathcal{A})' \longrightarrow \pi_{\varphi_n}(\mathcal{A})' \subseteq \mathcal{B}\left(\widehat{\mathcal{H}}_n\right)$$

is surjective where $\left(\pi_{\varphi_n}, \widehat{\mathcal{H}}_n, p_n\right)$ is a minimal Stinespring dilation of φ_n. Then

$$\rho\left(\varphi_n(\mathcal{A})'\right) = \rho(\mathbb{C}I) = \mathbb{C}I$$

by Lemma 2. It implies that $\pi_{\varphi_n}(\mathcal{A})' = \mathbb{C}I$ since ρ is surjective. Hence π_{φ_n} is irreducible and $p_n \in \pi_{\varphi_n}(\mathcal{A})''$. So, for these irreducible representation $\{\pi_{\varphi_n}\}$ of $\mathcal{A}$ on Hilbert space $\widehat{\mathcal{H}}_n$ and finite-rank projection $p_n \in \pi_{\varphi_n}(\mathcal{A})''$, we have $\|[p_n, \pi_{\varphi_n}(x)]\| \longrightarrow 0$ and $\limsup \|p_n \pi_{\varphi_n}(x) p_n\| = \|x\|$ for all $x \in \mathcal{A}$. It implies that $\mathcal{A}$ is inner QD by Theorem 2. $\square$

Suppose $\mathcal{A}$ is a unital C*-algebra and $p \in \mathit{socle}(\mathcal{A}^{**})$. Define

$$\mathcal{A}_p = \{a \in \mathcal{A} : [a, p] = 0\}.$$

Then we have the following few lemmas.

Lemma 3. *(*[5]*, Corollary 3.5.) Let* $p \in \mathit{socle}(\mathcal{A}^{**})$. *Then* $d(a, \mathcal{A}_p) = \|[a, p]\|$ *for all* $a \in \mathcal{A}$.

Lemma 4. *(*[5]*, Proposition 3.4.) Let* $\mathcal{A}$ *be a C*-algebra, and* $p \in \mathit{socle}(\mathcal{A}^{**})$. *Then*

(1) $p\mathcal{A}_p = p\mathcal{A}_p p = p\mathcal{A}^{**}p = p\mathcal{A}p$
(2) *The weak closure of* $\mathcal{A}_p$ *in* $\mathcal{A}^{**}$ *is* $p\mathcal{A}^{**}p + (1-p)\mathcal{A}^{**}(1-p)$.

Lemma 5. *Let* $\mathcal{A}$ *be a C*-algebra,* $p_1, \cdots, p_k \in \mathit{socle}(\mathcal{A}^{**})$ *with* $p_1 \leq \cdots \leq p_k$. *Then*

$$\begin{aligned} p_k\left(\cap_{i=0}^k \mathcal{A}_{p_i}\right) &= p_k\left(\cap_{i=0}^k \mathcal{A}_{p_i}^{**}\right) \\ &= p_0\mathcal{A}p_0 \oplus (p_1 - p_0)\mathcal{A}(p_1 - p_0) \oplus \cdots \oplus (p_k - p_{k-1})\mathcal{A}(p_k - p_{k-1}) \end{aligned}$$

Proof. We only prove the case when $k = 2$. Since $p_1 \leq p_2 \in \mathit{socle}(\mathcal{A}^{**})$, we have

$$p_2\left(\mathcal{A}_{p_1}^{**} \cap \mathcal{A}_{p_2}^{**}\right) = p_1\mathcal{A}p_1 + (p_2 - p_1)\mathcal{A}(p_2 - p_1)$$

by Lemma 4. So it is obvious that

$$p_2 \left(\mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}\right) \subseteq p_1 \mathcal{A} p_1 + (p_2 - p_1) \mathcal{A} (p_2 - p_1).$$

Meanwhile,

$$\begin{aligned} p_2 \mathcal{A}_{p_2} &= p_2 \mathcal{A} p_2 = p_2 \mathcal{A}^{**} p_2 \\ &\supseteq p_1 \mathcal{A} p_1 + (p_2 - p_1) \mathcal{A} (p_2 - p_1) \end{aligned}$$

by Lemma 4 and the fact that $p_1 \leq p_2 \in socle\left(\mathcal{A}^{**}\right)$. Then for every $b_1 \in p_1 \mathcal{A} p_1$ and $b_2 \in (p_2 - p_1) \mathcal{A} (p_2 - p_1)$,there is $a \in \mathcal{A}_{p_2}$ such that $p_2 a = b_1 + b_2$. Hence $a \in \mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}$. It implies that

$$p_2 \left(\mathcal{A}_{p_1} \cap \mathcal{A}_{p_2}\right) \supseteq p_1 \mathcal{A} p_1 + (p_2 - p_1) \mathcal{A} (p_2 - p_1).$$

This completes the proof. □

Lemma 6. *Let $\mathcal{A}$ be a C*-algebra, $\{p_n\}$ be a sequence of projection in $socle\left(\mathcal{A}^{**}\right)$ with $p_0 \leq p_1 \leq \cdots$ and $p_i \xrightarrow{s.o.t} I$ (strong operator topology). Then*

$$\cap_{i=0}^{\infty} \mathcal{A}_{p_i} = p_0 \mathcal{A} p_0 \oplus \oplus_{i=1}^{\infty} \left[(p_i - p_{i-1}) \mathcal{A} (p_i - p_{i-1})\right] \subseteq \mathcal{A}$$

Proof. By Lemma 5 and the fact that $p_1 \leq p_2 \leq \cdots$ with $p_i \in socle\left(\mathcal{A}^{**}\right)$, we have

$$\begin{aligned} p_k \left(\cap_{i=0}^{\infty} \mathcal{A}_{p_i}\right) &= p_k \left(\cap_{i=0}^{k} \mathcal{A}_{p_i}\right) = p_k \left(\cap_{i=0}^{k} \mathcal{A}_{p_i}^{**}\right) \\ &= p_0 \mathcal{A} p_0 \oplus (p_1 - p_0) \mathcal{A} (p_1 - p_0) \oplus \cdots \oplus (p_k - p_{k-1}) \mathcal{A} (p_k - p_{k-1}) \end{aligned}$$

for every k. Therefore

$$\begin{aligned} \cap_{i=0}^{\infty} \mathcal{A}_{p_i} &= p_0 \left(\cap_{i=0}^{\infty} \mathcal{A}_{p_i}\right) \oplus \oplus_{i=1}^{\infty} (p_i - p_{i-1}) \left(\cap_{k=0}^{\infty} \mathcal{A}_{p_k}\right) \\ &= p_0 \left(\cap_{i=0}^{\infty} \mathcal{A}_{p_i}\right) \oplus \oplus_{i=1}^{\infty} (p_i - p_{i-1}) \left(\cap_{k=0}^{i} \mathcal{A}_{p_k}\right) \\ &= p_0 \mathcal{A} p_0 \oplus \oplus_{i=1}^{\infty} \left[(p_i - p_{i-1}) \mathcal{A} (p_i - p_{i-1})\right] \end{aligned}$$

□

Remark 1. *Suppose $\mathcal{A}$ is a unital inner QD C*-algebras, then there is sequence $\{p_n\}$ of projections in $socle\left(\mathcal{A}^{**}\right)$ such that $\|[p_n, a]\| \longrightarrow 0$ for all $a \in \mathcal{A} \subseteq \mathcal{A}^{**}$ and $\|a\| = \lim \|p_n a p_n\|$ for all $a \in \mathcal{A}$ by Theorem 1. Therefore we can define a sequence of u.c.p maps $\varphi_n : \mathcal{A} \longrightarrow p_n \mathcal{A}^{**} p_n$ by compression. It is obvious that $\mathcal{A}_{p_n} = \mathcal{M}_{\varphi_n}$ where $\mathcal{M}_{\varphi_n}$ is the multiplicative domain of φ_n and $\|a\| = \lim \|\varphi_n(a)\|$, $d\left(a, \mathcal{M}_{\varphi_n}\right) \longrightarrow 0$ for all $a \in \mathcal{A}$ by Lemma 3. Actually, this is a sufficient condition for a given C*-algebra to be an inner QD C*-algebra.*

Theorem 5. *([8]) $\mathcal{A}$ is inner QD if and only if there is a sequence of c.c.p. maps $\varphi_n : \mathcal{A} \longrightarrow \mathcal{M}_{k_n}(\mathbb{C})$ such that $\|a\| = \lim \|\varphi_n(a)\|$ and $d\left(a, \mathcal{M}_{\varphi_n}\right) \longrightarrow 0$ for all $a \in \mathcal{A}$, where $\mathcal{M}_{\varphi_n}$ is the multiplicative domain of φ_n.*

Now, we are ready to give another characterization of unital inner QD C*-algebras.

Theorem 6. *Suppose $\mathcal{A}$ is a unital separable C*-algebra. Then $\mathcal{A}$ is inner QD if and only if there is a sequence of unital RFD C*-subalgebra $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $\mathcal{A}$ such that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is norm dense in $\mathcal{A}$.*

Proof. ($\Longrightarrow$) Suppose $\mathcal{F} \subseteq \mathcal{A}$ is a finite subset and $\varepsilon > 0$. Let

$$\{1\} \cup \mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$$

be the sequence of finite subsets of $\mathcal{A}$ such that $\overline{\cup_i \mathcal{F}_i} = \mathcal{A}$. Then, from Remark 1 and Theorem 5, we can find

$$P_0 \leq P_1 \leq P_2 \leq \cdots$$
$$\text{with } P_i \in socle\left(\mathcal{A}^{**}\right) \text{ and } P_i \overset{s.o.t}{\longrightarrow} P \in \mathcal{A}^{**} \ (i \longrightarrow \infty)$$

such that

$$\begin{aligned} d(a, \mathcal{A}_{P_i}) &= \|[a, P_i]\| \\ &= \|(1 - P_i)\, a P_i + P_i a\, (1 - P_i)\| < \frac{\varepsilon}{2 \cdot 2^{i+1}} \end{aligned}$$

and $\|P_i a P_i\| > \|a\| - \frac{\varepsilon}{2^{i+1}}$ for every $a \in \mathcal{F}_i$ $(i \in \mathbb{N})$. Since $P_i \overset{s.o.t}{\longrightarrow} P \in \mathcal{A}^{**}$ (as $i \longrightarrow \infty$) and $P \geq P_i$, we have

$$\|PaP\| \geq \|P_i a P_i\| \geq \|a\| - \frac{\varepsilon}{2^{i+1}} \text{ for } \forall a \in \cup_i \mathcal{F}_i \text{ and } i.$$

It implies that $\|PaP\| = \|a\|$ for $\forall a \in \overline{\cup_i \mathcal{F}_i} = \mathcal{A}$, therefore $P = I$. Now let

$$\mathcal{A}_\varepsilon = \cap_{i=0}^{\infty} \mathcal{A}_{P_i} = P_0 \mathcal{A} P_0 \oplus (P_1 - P_0)\, \mathcal{A}\, (P_1 - P_0) \oplus \cdots$$

by Lemma 6. So, for any $a \in \mathcal{F}$, let

$$x = P_0 a P_0 + (P_1 - P_0)\, a\, (P_1 - P_0) + \cdots \in \mathcal{A}_\varepsilon,$$

we have

$$\begin{aligned} d(a, \mathcal{A}_\varepsilon) &\leq \|a - x\| \\ &= ||P_0 a\, (P_1 - P_0) + P_1 a\, (P_2 - P_1) + \cdots \\ &+ (P_1 - P_0)\, a P_0 + (P_2 - P_1)\, a P_1 + \cdots || \\ &\leq \|P_0 a\, (1 - P_0)\|\, \|P_1 - P_0\| + \cdots + \|P_1 - P_0\|\, \|(1 - P_0)\, a P_0\| \\ &< \sum_{i=0}^{\infty} \frac{\varepsilon}{2 \cdot 2^{i+1}} + \sum_{i=0}^{\infty} \frac{\varepsilon}{2 \cdot 2^{i+1}} = \varepsilon. \end{aligned}$$

Note that $\mathcal{A}_\varepsilon$ is an RFD C*-subalgebras of $\mathcal{A}$, hence we can find a sequence of unital RFD C*-subalgebra $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $\mathcal{A}$ such that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is norm dense in $\mathcal{A}$.

($\Longleftarrow$) Suppose $\{\mathcal{A}_n\}_{n=1}^{\infty}$ is a sequence of unital RFD C*-subalgebra in $\mathcal{A}$ such that $\overline{\cup_{n=1}^{\infty} \mathcal{A}_n}^{\|\cdot\|} = \mathcal{A}$, $\mathcal{F} \subseteq \mathcal{A}$ is a finite subset and $\varepsilon > 0$. Then there is an RFD C*-subalgebra $\mathcal{A}_n$ and $b_a \in \mathcal{A}_n$ such that $\|a - b_a\| < \frac{\varepsilon}{3}$ for every $a \in \mathcal{F}$. It follows that

$$\|a\| - \frac{\varepsilon}{3} \leq \|b_a\|.$$

Since $\mathcal{A}_n$ is RFD, we can find a projection P such that $\Phi_P : \mathcal{A}_n \longrightarrow P \mathcal{A}_n P \subseteq \mathcal{M}_t(\mathbb{C})$ is a *-homorphism for some $t \in \mathbb{C}$ and $\|\Phi_P(b_a)\| \geq \|b_a\| - \frac{\varepsilon}{3}$. Extending Φ_P to a u.c.p. map $\widetilde{\Phi_P} : \mathcal{A} \longrightarrow \mathcal{M}_t(\mathbb{C})$ with $\mathcal{A}_n \subseteq \mathcal{M}_{\widetilde{\Phi_P}}$ and

$$\left\|\widetilde{\Phi_P}(b_a)\right\| \geq \|b_a\| - \frac{\varepsilon}{3}$$

where $\mathcal{M}_{\widetilde{\Phi_P}}$ is the multiplicative domain of $\widetilde{\Phi_P}$. Then

$$\left\|\widetilde{\Phi_P}(b_a)\right\| = \left\|\widetilde{\Phi_P}(b_a - a) + \widetilde{\Phi_P}(a)\right\| \leq \frac{\varepsilon}{3} + \left\|\widetilde{\Phi_P}(a)\right\| \text{ for every } a \in \mathcal{F}.$$

So from above inequalities, we have

$$\begin{aligned}\left\|\widetilde{\Phi_P}(a)\right\| &\geq \left\|\widetilde{\Phi_P}(b_a)\right\| - \frac{\varepsilon}{3} \\ &\geq \|b_a\| - \frac{2\varepsilon}{3} \geq \|a\| - \varepsilon \text{ for every } a \in \mathcal{F}\end{aligned}$$

By the preceding discussion, for a finite subset $\mathcal{F}$ and $\varepsilon > 0$, there is a u.c.p map $\widetilde{\Phi_P} : \mathcal{A} \longrightarrow \mathcal{M}_t(\mathbb{C})$ for some $t \in \mathbb{N}$ such that $d\left(a, \mathcal{M}_{\widetilde{\Phi_P}}\right) < \varepsilon$ and $\left\|\widetilde{\Phi_P}(a)\right\| \geq \|a\| - \varepsilon$ for every $a \in \mathcal{F}$. So by Theorem 5, $\mathcal{A}$ is inner. □

3. Unital Full Free Products of Two Inner QD Algebras.

In this section we will consider the question of whether the unital full free products of inner QD C*-algebras are inner QD again. First, we need a lemma for showing the main result in this section.

Lemma 7. *(Theorem 3.2,* [16]*) Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C*-algebras. Then the unital full free product $\mathcal{A} = \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is RFD if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both RFD.*

Now, we are ready to give the main result of this section.

Theorem 7. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both unital inner quasidiagonal C*-algebras. Then $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is inner QD.*

Proof. Suppose τ is a fixed state on $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ and $\mathcal{F}$ is a finite subset of $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$. Let

$$\mathcal{A}_j^0 = \{a \in \mathcal{A}_j : \tau(a) = 0\}, j = 1, 2.$$

No loss of generality, we may assume that every $b \in \mathcal{F}$ can be decomposed into a finite sum with respect to τ, that is

$$b = \alpha_0 I + \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} a_{i_1} a_{i_2} \cdots a_{i_n} \quad \alpha_0 \in \mathbb{C},\ a_{i_j} \in \mathcal{A}_{i_j}^0, i_1 \neq i_2 \neq \cdots \neq i_n$$

where $\mathcal{A}_{i_j}^0 = \mathcal{A}_1^0$ or $\mathcal{A}_2^0$. Denote by $\mathcal{F}_0^j$, $j = 1, 2$, the set of such elements of $\mathcal{A}_j^0$ which appear in the decomposition of elements from $\mathcal{F}$. Then we can find an RFD C*-subalgebra $\mathcal{A}_\varepsilon^j$ of $\mathcal{A}_j$ for $j = 1, 2$ such that

$$d(a, \mathcal{A}_\varepsilon^j) < \varepsilon \text{ for } \forall a \in \mathcal{F}_0^j, j = 1, 2. \tag{1}$$

Let b be an element in $\mathcal{F}$. No loss of generality, we may assume that b can be decomposed into the form

$$\alpha I + \sum_{i=1}^{l} a_{1,1}^i a_{2,1}^i a_{1,2}^i a_{2,2}^i a_{1,3}^i \cdots a_{1,n_i}^i,$$

where

$$\left\{a_{1,1}^i, a_{1,2}^i, \cdots, a_{1,n_i}^i\right\} \subseteq \mathcal{F}_0^1$$

and

$$\left\{a_{2,1}^i, a_{2,2}^i, \cdots, a_{2,n}^i\right\} \subseteq \mathcal{F}_0^2.$$

Then, for any $a_{j,k}^i$, we can find $\widetilde{a_{j,k}^i} \in \mathcal{A}_\varepsilon^j$ such that $\left\|a_{j,k}^i - \widetilde{a_{j,k}^i}\right\| < \varepsilon$ by (1). Note that

$$\left|\tau\left(a_{j,k}^i - \widetilde{a_{j,k}^i}\right)\right| = \left|\tau\left(a_{j,k}^i\right) - \tau\left(\widetilde{a_{j,k}^i}\right)\right| = \left|\tau\left(\widetilde{a_{j,k}^i}\right)\right| < \varepsilon,$$

then $\widetilde{a^i_{j,k}} = \tau\left(\widetilde{a^i_{j,k}}\right) + \left(\widetilde{a^i_{j,k}} - \tau\left(\widetilde{a^i_{j,k}}\right)\right)$ and $\left\|\left(\widetilde{a^i_{j,k}} - \tau\left(\widetilde{a^i_{j,k}}\right)\right) - a^i_{j,k}\right\| < 2\varepsilon$. Therefore, no loss of generality, we may assume that $\widetilde{a^i_{1,k}} \in \left(\mathcal{A}^1_\varepsilon\right)^0$ with $\left\|a^i_{1,k} - \widetilde{a^i_{1,k}}\right\| < 2\varepsilon$ where $k = 1, \cdots, n_i, i = 1, \cdots, l$. And $\widetilde{a^i_{2,k}} \in \left(\mathcal{A}^2_\varepsilon\right)^0$ with $\left\|a^i_{2,k} - \widetilde{a^i_{2,k}}\right\| < 2\varepsilon$ where $k = 1, \cdots, n_i, i = 1, \cdots, l$. Let $\widetilde{b} = \alpha I + \sum_{i=1}^l \widetilde{a^i_{1,1}}\widetilde{a^i_{2,1}}\widetilde{a^i_{1,2}} \cdots \widetilde{a^i_{1,n_i}} \in \mathcal{A}^1_\varepsilon *_{\mathbb{C}} \mathcal{A}^2_\varepsilon$. There is an integer $M_b > 0$ such that

$$\begin{aligned}
&\left\|b - \widetilde{b}\right\| \\
&= \left\|\alpha I + \sum_{i=1}^l a^i_{1,1}a^i_{2,1}a^i_{1,2}a^i_{2,2}a^i_{1,3} \cdots a^i_{1,n_i} - \left(\alpha I + \sum_{i=1}^l \widetilde{a^i_{1,1}}\widetilde{a^i_{2,1}}\widetilde{a^i_{1,2}} \cdots \widetilde{a^i_{1,n_i}}\right)\right\| \\
&\le M_b\varepsilon.
\end{aligned}$$

Since $\mathcal{A}^1_\varepsilon *_{\mathbb{C}} \mathcal{A}^2_\varepsilon$ is an RFD C*-algebra by Lemma 7, then by Theorem 6 we have that $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is inner QD. □

Since every strong NF algebra is inner, then from [4], we know that every AF algebra and AH algebra are inner. Hence we have the following two corollaries.

Corollary 1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are both AF algebras, then $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ is an inner QD algebra.*

Corollary 2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are both AH algebras, then $\mathcal{A} *_{\mathbb{C}} \mathcal{B}$ is an inner QD algebra.*

What will happen when the above amalgamation is over some other C*-algebras instead of $\mathbb{C}I$? In [13], it has been shown that a full amalgamated free product of two QD algebras may not be MF again, even for a unital full free product of two full matrix algebras with amalgamation over a two dimensional C*-algebra which is *-isomorphic to $\mathbb{C} \oplus \mathbb{C}$. Therefore, a unital full amalgamated free product of two unital inner QD C*-algebras may not be inner again. But we can give the affirmative answers for some specific cases.

The following result can be found in [15] or [13].

Lemma 8. *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are unital C*-algebras. Then*

$$(\mathcal{A}\otimes_{\max}\mathcal{D}) \underset{\mathcal{D}}{*} (\mathcal{B}\otimes_{\max}\mathcal{D}) \cong \left(\mathcal{A}\underset{\mathbb{C}}{*}\mathcal{B}\right) \otimes_{\max} \mathcal{D}.$$

Lemma 9. *([5]) Let $\mathcal{A}$ be a C*-algebra. Then, for any k, $\mathcal{A}$ is inner QD if and only if $\mathcal{M}_k(\mathcal{A}) = \mathcal{A} \otimes \mathcal{M}_k(\mathbb{C})$ is inner QD.*

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. If $\mathcal{D}$ can be embedded as an unital C*-subalgebra of $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\mathcal{D}$ is *-isomorphic to a full matrix algebra $\mathcal{M}_n(\mathbb{C})$ for some integer n, then the unital full amalgamated free product $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}$ is inner QD if $\mathcal{A}$ and $\mathcal{B}$ are both inner QD.*

Proof. Since $\mathcal{D}$ is *-isomorphic to a full matrix algebra, from Lemma 6.6.3 in [12], it follows that $\mathcal{A} \cong \mathcal{A}' \otimes \mathcal{D}$ and $\mathcal{B} \cong \mathcal{B}' \otimes \mathcal{D}$. Then $\mathcal{A}'$ and $\mathcal{B}'$ are inner QD by Lemma 9. So the desired conclusion follows from Theorem 7, Lemma 8 and Lemma 9. □

Next, we will consider the case when the free products are amalgamated over some finite-dimensional C*-algebras.

Lemma 10. *([6]) An arbitrary inductive limit (with injective connecting maps) of inner quasidiagonal C*-algebras is inner quasidiagonal.*

Lemma 11. *([15], Theorem 4.2) Assume that we have embeddings of C*-algebras $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{C} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2$, then the natural morphism $\sigma : \mathcal{A}_1 *_{\mathcal{C}} \mathcal{B}_1 \longrightarrow \mathcal{A}_2 *_{\mathcal{C}} \mathcal{B}_2$ is injective.*

Lemma 12. *([15] corollary 4.13) If $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ are increasing sequences of C*-algebras, all of which contain a common C*-subalgebra $\mathcal{C}$, then there is a natural isomorphism*

$$\varinjlim (\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n) = \varinjlim \mathcal{A}_n *_{\mathcal{C}} \varinjlim \mathcal{B}_n$$

where $\varinjlim$ denotes the ordinary direct limit.

The following lemma is a well-known property of AF algebras, we can find it in [9]

Lemma 13. *A C*-algebra $\mathcal{A}$ is AF if and only if it is separable and*

$(*)$ for all $\varepsilon > 0$ and $A_1, \cdots, A_n$ in $\mathcal{A}$, there exists a finite dimensional C-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that dist $(A_i, \mathcal{B}) < \varepsilon$ for $1 \leq i \leq n$.*

Moreover, if $\mathcal{A}_1$ is a finite-dimensional subalgebra of $\mathcal{A}$, then we may choose $\mathcal{B}$ so that it contains $\mathcal{A}_1$.

Lemma 14. *([1], Theorem 4.2) Consider unital inclusions of C*-algebras $\mathcal{A} \supseteq \mathcal{C} \subseteq \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ finite dimensional. Let $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ be the corresponding full amalgamated free product. Then $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ is RFD if and only there are faithful tracial states $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and $\tau_{\mathcal{B}}$ on $\mathcal{B}$ with*

$$\tau_{\mathcal{A}}(x) = \tau_{\mathcal{B}}(x), \qquad \forall\ x \in \mathcal{C}.$$

Corollary 3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are AF algebras and $\mathcal{A} \supseteq \mathcal{C} \subseteq \mathcal{B}$ with $\mathcal{C}$ finite-dimensional. If there are faithful tracial states $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$ respectively, such that*

$$\tau_{\mathcal{A}}(x) = \tau_{\mathcal{B}}(x), \qquad \forall\ x \in \mathcal{C},$$

then $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ is inner QD..

Proof. Since $\mathcal{C}$ is a finite-dimensional C*-subalgebra, then we can find a sequence of finite-dimensional C*-subalgebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ and $\{\mathcal{B}_n\}_{n=1}^{\infty}$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ with $\overline{\cup \mathcal{A}_n} = \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots$ with $\overline{\cup \mathcal{B}_n} = \mathcal{B}$ by Lemma 13. Note that $\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n$ is RFD by Lemma 14, then $\mathcal{A} *_{\mathcal{C}} \mathcal{B} = \varinjlim (\mathcal{A}_n *_{\mathcal{C}} \mathcal{B}_n)$ is inner by Lemma 12 and Lemma 10. □

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